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Stability and Accuracy of Spatial Approximations for Wave Equation Tidal Models

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Amplitude and phase characteristics for numerical approximations to the shallow water wave equation are obtained for linear and quadratic finite elements, for finite difference approximations, for non-constant bathemetry, and for uneven node spacing. Stability is shown to require non-zero friction as well as satisfaction of a Courant constraint. Lumping is shown to reduce the Courant constraint for stability while higher order and quadratic finite element approximations require a more restrictive constraint than their second order and linear finite element counterparts. The amplitude of the propagation factor for stable schemes and propagating waves is seen to be independent of the Courant number and type of numerical approximation. Although the higher order and quadratic schemes provide better propagation of the low and moderate frequency waves, the highest frequency waves ($2\Delta x$) are better propagated by low order numerical methods.

INTRODUCTION

Stability and accuracy properties of finite element schemes for the solution of differential equations provide tremendous insight into the expected behavior of a computational scheme. Gray and Lynch [3] have studied these properties for various time marching schemes for solution of the linearized 1-dimensional shallow water equations. Their analysis was restricted to the case of linear elements and constant bathymetry. They showed that solution procedures for the primitive shallow water equations found to be effective in finite difference modeling will be plagued by node-to-node, or $2\Delta x$, oscillations in finite element modeling without artificial viscosity. However, a wave equation formulation of the continuity equation overcomes these difficulties because of its ability to propagate the $2\Delta x$ oscillations (Lynch and Gray [8]).

Kinnmark and Gray [5] have recently analyzed the wave equation scheme in two space dimensions including the Coriolis force. They considered both conditionally stable explicit schemes and an unconditionally stable implicit scheme. Their implicit algorithm requires that a symmetric matrix be solved for the wave continuity equations while the momentum equation only requires the solution of a simple diagonal matrix.

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Because earlier studies have demonstrated the inferiority of many primitive equation schemes compared to the wave equation formulation (Lynch and Gray, [8]; Gray, [1]), these schemes will no be studied further here. However Kinnmark [4] has done a detailed complementary study to the current presentation which further verifies the clear superiority of the wave equation approach. The present study determines the stability conditions and accuracy properties of the explicit wave equation scheme for quadratic Lagrangian elements in one dimension. The application of these elements to 2-dimensional problems has met with some success (Gray and Kinnmark, [2]; Laible, [6]), however, an appropriate numerical analysis has been lacking heretofore. For the 2-dimensional case with equal node spacing and constant bathymetry when the Coriolis force is neglected, the stability constraint is identical in form to the 1-dimensional case with the square of the Courant number replaced by the sum of squares of the Courant number in the x and y directions.

One of the important basic features of finite elements is the ease with which non-uniform grids may be implemented. However the impact of grid non-uniformities on stability and accuracy is rarely given theoretical attention. In the present analysis, a simple periodic grid consisting of two different alternating mesh widths is chosen for closer investigation. It is shown that stability for the wave equation scheme, using quadratic elements, may be obtained only if the ratio of the large-to-small mesh width is less than three. Finally the effect of variable bathymetry on stability is also considered. Herein the bathymetry is allowed to vary periodically with alternating depths appearing at adjacent node points. The stability constraint is shown to depend on both depths.

Equations Considered

This study focuses on alternative procedures for solving the vertically integrated equations describing shallow water flow. For the numerical analysis here, the linearized forms will be considered. The primitive formulations of these equations for conservation of mass is

$$L(\zeta, U, V) \equiv \frac{\partial \zeta}{\partial t} + h\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0 \tag{1}$$

The 2-dimensional vector momentum equation has the scalar components

$$M_x(\zeta, U, V) \equiv \frac{\partial U}{\partial t} + \tau U - fV + g\frac{\partial \zeta}{\partial x} = 0 \tag{2}$$

and

$$M_y(\zeta, U, V) \equiv \frac{\partial V}{\partial t} + \tau V + fU + g\frac{\partial \zeta}{\partial y} = 0 \tag{3}$$

where

ζ	is elevation above a datum,
U	is the vertically averaged eastward velocity,
V	is the vertically averaged northward velocity,
h	is the bathymetry,
τ	is the linearized bottom friction parameter,
f	is the Coriolis parameter,
g	is gravitational acceleration,
x	is positive eastward,
y	is positive northward, and
t	is time.

Lynch and Gray [8] used a wave equation in place of Eq. (1). This equation may be derived from (1)–(3) as

$$W(\zeta, U, V) = \frac{\partial L}{\partial t} + \tau L - h\frac{\partial M_x}{\partial x} - h\frac{\partial M_y}{\partial y} = 0 \tag{4}$$

or

$$W(\zeta, U, V) = \frac{\partial^2 \zeta}{\partial t^2} + \tau\frac{\partial \zeta}{\partial t} - gh\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) + fh\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) = 0. \tag{5}$$

It has been demonstrated (Lynch and Gray, [8]) that replacement of (1) by (5) provides a very significant decrease in the node-to-node, or $2\Delta x$, oscillations typically encountered in a primitive equation scheme.

Temporal and Spatial Approximations

The remainder of the paper will be concerned with comparisons of different temporal and spatial approximations to the momentum equations (2) and (3) in conjunction with the wave-continuity equation (5). Spatial approximation using both consistent and lumped coefficient matrices generated from linear and quadratic isoparametric Lagrangian finite elements as well as second- and fourth-order accurate finite difference approximations will be considered. When desired, the lumping is obtained automatically by use of nodal quadrature rather than Gaussian or exact integration. The trapezoidal rule with linear elements and Simpson's rule with quadratic elements in local coordinates both give rise to lumped matrices.

For the second order in time wave equations, a simple three-time-level symmetric approximation is used in the discretization. For the first-order momentum equations, a Crank–Nicolson two-level discretization is applied.

Stability Considerations

Upon application of either a finite difference or finite element procedure, discrete approximations to the differential equations under study are obtained. These approximations are combinations of the unknown coefficients which are estimates of the solution at a node. If the index j is used to indicate position on the x axis and k indicates position in time such that $\zeta_{j,k} \approx \zeta(j\Delta x, k\Delta t)$ where Δx and Δt are constant increments, then, for example, $\partial^2\zeta/\partial x^2$ might be approximated as

$$\frac{\partial^2\zeta}{\partial x^2} \approx \frac{\zeta_{j+1,k} - 2\zeta_{j,k} + \zeta_{j-1,k}}{\Delta x^2} \tag{6}$$

The study here is designed to investigate the growth or decay of the solution at a particular nodal location with time. The solution may be viewed as a combination of Fourier components such that, for example, in one dimension

$$\zeta = \sum \zeta_0 e^{\hat{\imath}\beta_m t + \hat{\imath}\sigma_m x}, \tag{7a}$$

$$U = \sum U_0 \, e^{\hat{\imath}\beta_m t + \hat{\imath}\sigma_m x}, \tag{7b}$$

and $\hat{\imath} = \sqrt{-1}$. Because the equations being analyzed are linear, it is possible to analyze each component of the Fourier sum separately. For simplicity the following notation will also be used for a representative component.

$$\zeta_{j,k} = \zeta_0 e^{\hat{\imath}\beta k \Delta t + \hat{\imath}\sigma j \Delta x} = \zeta_k e^{\hat{\imath}\sigma j \Delta x}. \tag{8}$$

Incorporation of this notation into Eq. (6) allows the spatial dependence to be accounted for completely through the $e^{\hat{\imath}\sigma\Delta x}$ term as

$$\frac{\zeta_{j+1,k} - 2\zeta_{j,k} + \zeta_{j-1,k}}{\Delta x^2} = \frac{e^{\hat{\imath}\sigma\Delta x} - 2 + e^{-\hat{\imath}\sigma\Delta x}}{\Delta x^2}\zeta_{j,k} = \frac{2(\cos\sigma\Delta x - 1)}{\Delta x^2}\zeta_{j,k}. \tag{9}$$

Similarly with all other terms that arise in the discrete approximation, all spatial dependence can be accounted for by use of the Fourier expression. Thus the 1-dimensional discretized form of Eq. (5) for node j with Coriolis forces neglected can be expressed as

$$\sum_{i=0}^{N} \alpha_i \zeta_{j,k+i} = 0, \tag{10}$$

where α_i depend on $\sigma\Delta x$ and are real, and $N+1$ is the number of time levels involved in the discretization. The solution to (10) is thus determined by the complex roots of the Nth degree polynomial

$$\sum_{i=0}^{N} \alpha_i \lambda^i = 0, \tag{11}$$

where λ equals $e^{i\beta \Delta t}$. It can be shown that solutions to the 2-dimensonal equations as well as equations which contains U, V, and ζ as unknowns are all dependent upon an equation of the same form as (11) although $N+1$ will not necessarily equal the number of time levels.

The roots of Eq. (11) are denoted by

$$\lambda_1, \ldots, \lambda_N,$$

and it is assumed that each λ_i is distinct such that the solution for ζ_k in Eq. (10) is

$$\zeta_k = \sum_{i=1}^{N} S_i(k\, \Delta t) = \sum_{i=1}^{N} C_i(\lambda_i)^k. \tag{12}$$

Now examine the mathematical properties of the lth root of this series. If $|\lambda_l| < 1$, $|S_l|$ will clearly approach zero as k tends to infinity. This corresponds to a stable root. However if $|\lambda_l| > 1$, $|S_l|$ will tend to infinity as k becomes infinite. This latter case is called an exponential instability because $|S_l|$ depends on k exponentially. Finally if the roots $|\lambda_l| = 1$, then as k approaches a large value, $|S_l|$ will neither tend to infinity nor zero but will be equal to $|C_l|$. This a referred to as a neutrally stable case because the solution approached is neither zero nor infinity.

Now consider the case where the roots λ_i may be double roots. (Higher order multiplicity gives qualitatively the same results.) If λ_l is a double root, the solution S_l associated with this root will be of the form

$$S_l(k\, \Delta t) = [C_l + kD_l]\, \lambda_l^k. \tag{13}$$

If $|\lambda_l| \neq 1$, stability or exponential instability is determined as above depending upon whether $|\lambda_l|$ is less or greater than 1. However for the case $|\lambda_l| = 1$, the implications are quite different. Due to the presence of the term kD_l in (13) the magnitude of S_l will tend toward infinity at large k provided $D_l \neq 0$. This last case will be called an instability with polynomial growth because $|S_l|$ depends on k.

In analyzing a numerical scheme, the problem then is to determine that neither exponential nor polynomial instabilities exist. In other words, the roots λ_i of the polynomial equations governing the system must all have magnitude less than one; or if roots of magnitude equal to one exist, these roots must be non-multiple. Kinnmark [4] has investigated the criteria that ensure the stability of Eq. (11) for N up to 4. However for those equations under study here, it is only necessary to examine stability for quadratic equations of the form

$$\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 = 0. \tag{14}$$

Stability for this equation such that both roots have magnitude less than 1 is conveniently decided upon by applying the Liénard–Chipart modification of the

TABLE I

Character of Roots of a Second-Degree Polynomial which is Not Exponentially Unstable

Conditions	Roots	Comment
$p_1 p_2 > 0$; $p_0 = 0$	$\lambda_1 = -1$; $\|\lambda_2\| < 1$	Stable
$p_0 p_2 > 0$; $p_1 = 0$	$\|\lambda_1\| = \|\lambda_2\| = 1$	Stable (λ_1 and λ_2 are complex conjugates.)
$p_0 p_1 > 0$; $p_2 = 0$	$\lambda_1 = +1$; $\|\lambda_2\| < 1$	Stable
$p_2 \neq 0$; $p_0 = p_1 = 0$	$\lambda_1 = \lambda_2 = -1$	Polynomial instability
$p_1 \neq 0$; $p_0 = p_2 = 0$	$\lambda_1 = -1$; $\lambda_2 = 1$	Stable
$p_0 \neq 0$; $p_1 = p_2 = 0$	$\lambda_1 = \lambda_2 = +1$	Polynomial instability
$p_0 = p_1 = p_2 = 0$	None (equation vanishes)	$\alpha_0 = \alpha_1 = \alpha_2 = 0$ (trivial)

Routh–Hurwitz criteria to the coefficients α_i (Porter, [9]). This procedure is used to define

$$p_0 = \alpha_0 - \alpha_1 + \alpha_2 \tag{15a}$$

$$p_1 = 2(\alpha_2 - \alpha_0) \tag{15b}$$

$$p_2 = \alpha_0 + \alpha_1 + \alpha_2. \tag{15c}$$

The necessary and sufficient conditions to guarantee $|\lambda_1| < 1$ and $|\lambda_2| < 1$ is that $p_0 p_1 > 0$, $p_0 p_2 > 0$, and $p_1 p_2 > 0$. If any of these products is negative, the scheme under study is exponentially unstable. If any of the p_i's is zero, but all $p_i p_j$ products are non-negative, a neutrally stable or polynomially unstable root exists. These possibilities are summerized in Table I. From examination of this table, the following conclusion can be drawn:

> If the coefficients α_i of a second degree polynomial describing a numerical solution algorithm are such that all products $p_i p_j$, where p_i is defined in (15), are nonnegative and at least one p_i is non-zero, the algorithm will be stable unless $p_1^2 = p_0 p_2 = 0$ in which case the algorithm will be polynomially unstable.

With this general tool established, some specific numerical algorithms will be examined for stability.

Applications with Even Node Spacing

The techniques discussed previously are very valuable in determining stability constraints for numerical solution of the shallow water equations. The actual application can be somewhat tedious in that a difference equation must be obtained and manipulated for each special case. In this sections the particular analysis for solution of the 1-dimensional wave and momentum equations on a quadratic finite element grid will be presented. The integration formula used will be Simpson's rule so that the approximation is lumped. Further the nodal spacing will be assumed constant. Stability criteria for other schemes will also be presented subsequently although the computational details are omitted.

For the 1-dimensional wave equation analysis with no Coriolis forces, the equations to be analyzed are obtained from (5) and (2) as

$$\mathscr{W}(\zeta, U) \equiv \frac{\partial^2 \zeta}{\partial t^2} + \tau \frac{\partial \zeta}{\partial t} - gh \frac{\partial^2 \zeta}{\partial x^2} = 0 \tag{16}$$

and

$$\mathscr{M}(\zeta, U) \equiv \frac{\partial U}{\partial t} + \tau U + g \frac{\partial \zeta}{\partial x} = 0. \tag{17}$$

These equations are to be solved by the Galerkin finite element method using a uniform grid. Both U and ζ are expanded in terms of quadratic Lagrangian basis functions and the approximations are substituted back into (16) and (17). Application of the Galerkin procedure yields:

$$\int_x \mathscr{W}(\hat{\zeta}, \hat{U})\, \phi_l \, dx = 0, \tag{18}$$

$$\int_x \mathscr{M}(\hat{\zeta}, \hat{U})\, \phi_l \, dx = 0, \tag{19}$$

where the $\hat{}$ indicates the approximation is used and ϕ_l is the basis function associated with node l. The integrated expressions obtained will be different depending on whether l is an interior node or a node on the boundary of an element. The approximations obtained for the wave equation by Simpson's rule integration are

$$l = j, \qquad \frac{d^2 \zeta_j}{dt^2} + \tau \frac{d\zeta_j}{dt} - gh \frac{\zeta^*_{j+1} - 2\zeta_j + \zeta^*_{j-1}}{\Delta x^2} = 0 \tag{20a}$$

$l = j + 1$,

$$\frac{d^2 \zeta^*_{j+1}}{dt^2} + \tau \frac{d\zeta^*_{j+1}}{dt} - gh \frac{-2\zeta^*_{j-1} + 16\zeta_j - 28\zeta^*_{j+1} + 16\zeta_{j+2} - 2\zeta^*_{j+3}}{8\Delta x^2} = 0 \tag{20b}$$

where ζ and ζ^* indicate values of surface elevation at intra- and interelement nodes, respectively. Use of Fourier expansions in space and finite differencing in time for the explicit case yield

$$\frac{\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}}{\Delta t^2} + \tau \frac{\zeta_{k+1} - \zeta_{k-1}}{2\Delta t} + \frac{2gh}{\Delta x^2} \zeta_k - \frac{2\cos(\sigma \Delta x)\, gh}{\Delta x^2} \zeta^*_k = 0, \tag{21a}$$

$$\frac{\zeta^*_{k+1}-2\zeta^*_k+\zeta^*_{k-1}}{\Delta t^2}+\tau\frac{\zeta^*_{k+1}-\zeta^*_{k-1}}{2\Delta t}-\frac{4gh\cos(\sigma\Delta x)}{\Delta x^2}\zeta_k$$
$$+\frac{gh(3+\cos^2(\sigma\Delta x))}{\Delta x^2}\zeta^*_k=0, \qquad (21b)$$

where the subscript k refers to a position in time and the spatial subscript j is understood. Using $\lambda=e^{i\beta\Delta t}$ such that, for example, $\zeta_{k+1}=\lambda\zeta_k$ and eliminating ζ_k and ζ^*_k between (21a) and (21b) one obtains

$$A^2+[6-\sin^2(\sigma\,\Delta x)]\,\mathscr{C}^2A\lambda+[6\sin^2(\sigma\,\Delta x)]\,\mathscr{C}^4\lambda^2=0, \qquad (22)$$

where

$$A=(\lambda-1)^2+(\tau\,\Delta t/2)(\lambda^2-1),$$

and

$$\mathscr{C}=\text{Courant number}=\frac{\Delta t\sqrt{gh}}{\Delta x}.$$

Although (22) is a fourth degree polynomial, it may be factored into a product of two second degree polynomials such that

$$[A+\gamma_+\lambda\mathscr{C}^2][A+\gamma_-\lambda\mathscr{C}^2]=0, \qquad (23)$$

where

$$\gamma_\pm=\tfrac{1}{2}\{6-\sin^2(\sigma\,\Delta x)\pm\sqrt{[6-\sin^2(\sigma\,\Delta x)]^2-24\sin^2(\sigma\,\Delta x)}\}. \qquad (24)$$

The four roots of Eq. (23) are made up of two roots which correspond to the approximate analytical solution plus two roots which are numerical artifacts and do not correspond to the physical problem. The desirable roots arise from the part of (23) containing γ_- while the extraneous roots are associated with the γ_+ portion. It is possible to obtain stability constraints associated with all of these roots using the technique described previously.

Expansion of the polynomial $[A+\gamma_\pm\lambda\mathscr{C}^2]$ yields

$$\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2=0, \qquad (25a)$$

where

$$\alpha_0=1-\frac{\tau\,\Delta t}{2}, \qquad (25b)$$

$$\alpha_1=-2+\mathscr{C}^2\gamma_\pm, \qquad (25c)$$

$$\alpha_2=1+\frac{\tau\,\Delta t}{2}. \qquad (25d)$$

To examine stability, the p's are calculated by Eq. (15) as

$$p_0 = 4 - \mathscr{C}^2 \gamma_{\pm} \tag{26a}$$

$$p_1 = 2\tau \, \Delta t \tag{26b}$$

$$p_2 = \mathscr{C}^2 \gamma_{\pm}. \tag{26c}$$

When friction is non-zero, $\mathscr{C}^2$, $\tau \, \Delta t$, and $\gamma_{\pm}$ are all non-negative; and the stability constraint is

$$4 > \mathscr{C}^2 \gamma_{\pm}. \tag{27}$$

The maximum value of γ_- is 2 while the maximum of γ_+ is 6. Therefore stability is controlled by the roots associated with the artifact and the constraints are

$$\tfrac{2}{3} > \mathscr{C}^2 \qquad \text{and} \qquad \tau \, \Delta t > 0. \tag{28}$$

It is interesting to note that if the numerical artifacts were eliminated, the strictness of the stability constraint would be reduced by a factor of 3 and would be less stringent than that for linear elements.

For the $2 \Delta x$ wave, $\sigma \, \Delta x = \pi$, and thus $\gamma_- = 0$. Therefore when $\tau \, \Delta t = 0$, indicating a simulation in the absence of friction, the criteria indicating polynomial instability ($p_1^2 = p_0 p_2 = 0$) are satisfied. This means that a stable simulation for the quadratic elements in the absence of friction is impossible. In fact the polynomial instability was found to be a difficulty for all the wave equation schemes with $\tau \, \Delta t = 0$. Stability criteria obtained for the various 1-dimensional discretizations examined appear in Table II.

Besides considering the stability properties of a numerical scheme, it is also important to determine its accuracy. This is accomplished through a comparison of λ obtained from a discretized equation with λ of the analytical solution. The complex propagation factor T (Leendertse, [7]) defined as the ratio of the numerical to analytic λ raised to the Nth power where N is the number of steps of size Δt required for the analytic wave to propagate one wavelength, provides a measure of accuracy. The magnitude of T is ideally 1 and the decrease (increase) from unity is an indication of the excessive (insufficient) damping properties of the scheme. The ideal phase of T is zero and inability of the scheme to propagate the wave at the correct velocity is indicated by a phase lag or lead. The expressions for λ for each of the six schemes under study appear in Table III. The modulus and phase of the propagation factors for these schemes are presented in Fig. 1 and 2.

Perhaps the most surprising aspect of Fig. 1 is that the moduli of the propagation factors are identical for all schemes over most of the range of the abscissa. Reference to the expression for λ in Table III indicates that when the quantity under the radical sign is negative such that λ is complex,

$$|\lambda| = \sqrt{(1-F)/(1+F)} \tag{29}$$

TABLE II

Stability Criteria for Explicit Wave Equation Methods Using Different Spatial Discretization on a Uniform One-Dimensional Grid with Constant Bathymetry

Spatial Approximation	Consistent linear finite element (CL)	Lumped linear finite element (LL)	Consistent quadratic finite element (CQ)	Lumped quadratic finite element (LQ)	Second-order finite difference (FD2)	Fourth-order finite difference (FD4)
Stability criteria	$\frac{1}{3} > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$1 > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$\frac{4}{15} > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$\frac{2}{3} > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$1 > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$\frac{3}{4} > \mathscr{C}^2$ and $\tau\,\Delta t > 0$

TABLE III

λ Values for Equi-spaced Grid Schemes

Spatial approximations		ω
Consistent linear finite element	(CL)	$3(1-c)/(2+c)$
Lumped linear finite element	(LL)	$1-c$
Consistent quadratic finite element	(CQ)	$[11+4c^2 \pm \sqrt{9(3+2c^2)^2-80(1-c^2)^2}]/[8-4c^2]$
Lumped quadratic finite element	(LQ)	$[5+c^2 \pm \sqrt{(1+c^2)^2+32c^2}]/4$
Second-order finite difference	(FD2)	$1-c$
Fourth-order finite difference	(FD4)	$(1-c)(7-c)/6$

Note. $\lambda = (1-\mathscr{C}^2\omega \pm \sqrt{(1-\mathscr{C}^2\omega)^2-(1+F)(1-F)})/(1+F)$, $\mathscr{C}$ = Courant number, $F=\tau\,\Delta t/2$, $c=\cos(\sigma\,\Delta x)$.

for all schemes and independent of the Courant number and wavelength. The variation in $|T|$ at small abscissa in Fig. 1 occur because the quantity under the radical in the expression for λ is positive in this range. A positive quantity under the radical sign in the expression for λ also causes λ to be purely real. Therefore if λ is positive the phase-velocity is zero and no propagation occurs. It can also be deduced that any stable scheme which produces a nonzero ω, for a given

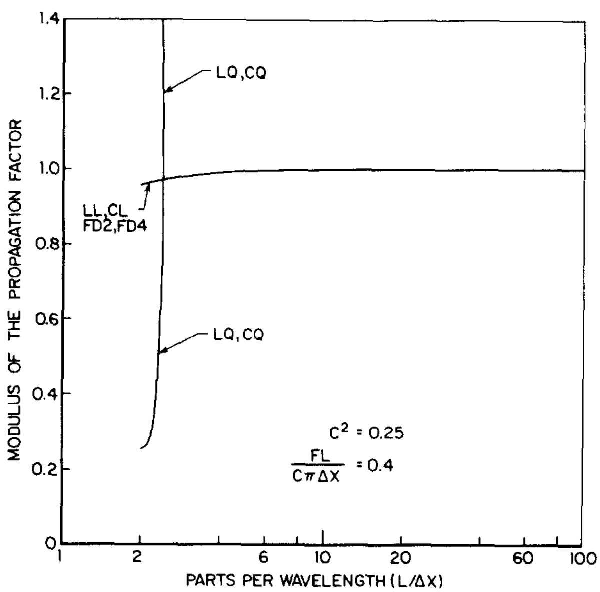

FIG. 1. Modulus of the propagation factor for the schemes of Table II.

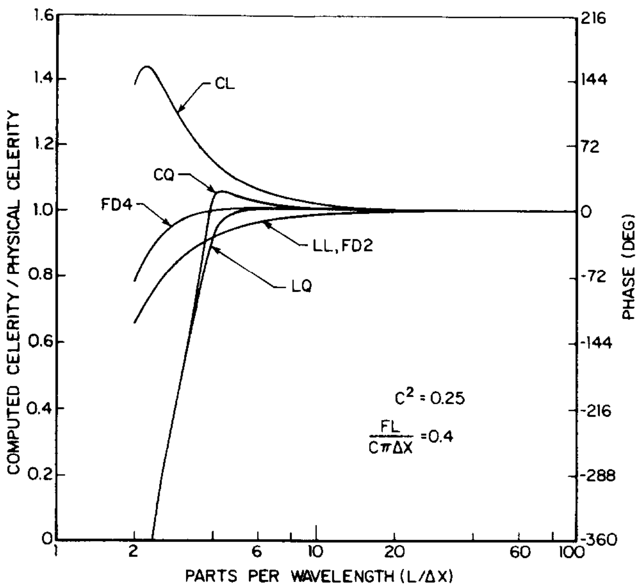

FIG. 2. Phase of the propagation factor for the schemes of Table II.

wavelength, will always propagate that wavelength if $F=0$. Figure 2 indicates some variability in the abilities of the various schemes to propagate waves. The FD4 scheme seems to be the best over the entire range of values of $L/\Delta x$. In the range $4\,\Delta x < L < 20\,\Delta x$, the quadratic finite element schemes are superior to the linear finite element schemes. For $L > 20\,\Delta x$, all methods are roughly comparable.

Application to Unequal Node Spacing

In this section, stability restrictions on wave equation (5) are examined for unequal node spacing. Because finite elements especially, but also some finite difference procedures, are frequently used on irregular meshes, the questions of how the variable mesh affects accuracy and stability are quite important. For simplicity a grid which alternates between two different mesh sizes will be examined. Stability constraints obtained appear in Table IV. The consistent linear finite element scheme and a generalization of the fourth-order finite difference scheme to a third-order scheme on an irregular grid, FD4, were analyzed. Their stability constraints were obtained following the procedures discussed previously. From Table IV it can be seen that the stability constraints for these schemes depend on an average Courant number in the sense of a geometric mean. The final scheme analyzed on an irregular grid was a quadratic finite element scheme making use of the standard isoparametric transformation prior to integration by Simpson's rule. This gives rise

TABLE IV

Stability Criteria for Explicit Wave Equation Methods Using Unequal Spatial Grid Steps in One Dimension with Constant Bathymetry

Spatial approximation	Second-order finite difference (FD2)	Lumped linear finite element (LL)	Consistent linear finite element (CL)	Fourth-order finite difference (FD4)	Lumped quadratic finite element (LQ)
Stability criteria	$1 > \mathscr{C}\mathscr{C}_\alpha$ and $\tau\,\Delta t > 0$	$1 > \mathscr{C}\mathscr{C}_\alpha$ and $\tau\,\Delta t > 0$	$\frac{1}{3} > \mathscr{C}\mathscr{C}_\alpha$ and $\tau\,\Delta t > 0$	$\frac{3}{4} > \frac{(3\alpha+3)}{(2\alpha+4)}\frac{(3\alpha+3)}{(4\alpha+2)}\mathscr{C}\mathscr{C}_\alpha$ and $\tau\,\Delta t > 0$	$\frac{2}{3} > \frac{2}{(3-\alpha)}\frac{2}{(3\alpha-1)}\mathscr{C}^2$ and $3 > \alpha > \frac{1}{3}$ and $\tau\,\Delta t > 0$

Note. $\mathscr{C} = \Delta t\sqrt{gh}/\Delta x$. $\mathscr{C}_\alpha = \mathscr{C}/\alpha$.

TABLE V

λ Values for Unequally-spaced Grid Schemes

Spatial approximation	ω
Second-order finite difference (FD2)	$[1+\alpha \pm \sqrt{(1-\alpha)^2+4\alpha c^2}]/[\alpha(1+\alpha)]$
Lumped linear finite element (LL)	$[1+\alpha \pm \sqrt{(1-\alpha)^2+4\alpha c^2}]/[\alpha(1+\alpha)]$
Consistent linear finite element (CL)	$4[1-c^2]/[(\alpha+1)^2-\frac{2}{3}(\alpha^2+1)(1-c^2)$ $\pm(\alpha+1)\sqrt{(\alpha+1)^2+\frac{4}{9}(\alpha-1)^2(1-c^2)^2-\frac{4}{3}(\alpha^2+\alpha+1)(1-c^2)}]$
Fourth-order finite difference (FD4)	$\frac{1}{2}[I+(G+K)(2c^2-1)$ $\pm\sqrt{4c^2(c^2-1)(G-K)^2+H^2+J^2+2(2c^2-1)HJ}]$
Lumped quadratic finite element (LQ)	$\left[6-\left[1+8\left(\frac{\alpha-1}{\alpha+1}\right)^2\right][1-c^2]\right.$ $\left.\pm\sqrt{\left[1+8\left(\frac{\alpha-1}{\alpha+1}\right)^2\right][1-c^2]^2+36c^2}\right]\Big/[(3-\alpha)(3\alpha-1)]$

Note. $\lambda = (1-\mathscr{C}^2\omega \pm \sqrt{(1-\mathscr{C}^2\omega)^2-(1+F)(1-F)})/(1+F)$, $\mathscr{C} = \sqrt{gh}\,\Delta t/\Delta x$, $F = \tau\,\Delta t/2$, $c = \cos((\sigma\,\Delta x(1+\alpha))/2)$, $G = (1+\alpha-\alpha^2)/\alpha(\alpha+1)^2(2\alpha+1)$, $H = (2(\alpha+1))/(\alpha(\alpha+2))$, $I = 2(\alpha^2+3\alpha+1)/(\alpha(\alpha+1)^2)$, $J = 2(\alpha+1)/(\alpha(2\alpha+1))$, $K = (\alpha^2+\alpha-1)/(\alpha+1)^2(\alpha+2)$.

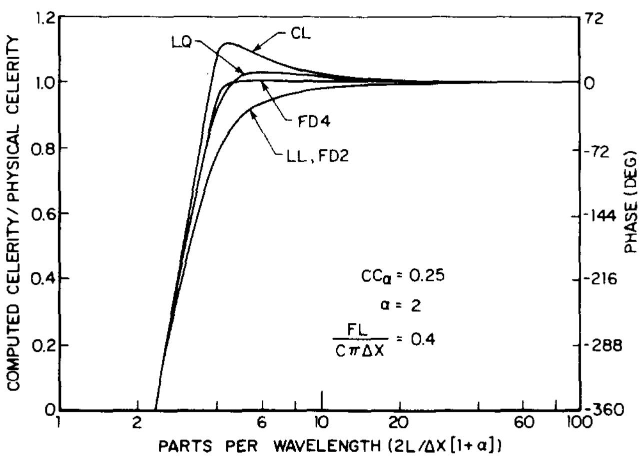

FIG. 3. Phase of the propagation factor for the schemes of Table IV.

to complete lumping of the time derivatives. Of note is the fact that the interior node must be in the center half of the element for stability (as well as uniqueness of the Jacobian).

The expressions for λ for the five schemes considered with unequal node spacing appear in Table V. As with the previous cases with even node specing, over most of the range of wavelengths the magnitude of λ is given by Eq. (29) and is independent of wavelength and Courant number as well as α.

Because the amplitude of the propagation factor for these cases is the same as for the even grid spacing, Fig. 1 should be referred to for $|T|$. The phase of T, however, does depend on α and the plot of the phase for $\alpha = 2$ appears as Fig. 3. The phase errors presented are seen to be somewhat worse than the corresponding errors in Fig. 2 which depict the $\alpha = 1$ cases. It is also important to realize that for the grid under study the minimum (non-aliased) wavelength is $(\alpha + 1)\,\Delta x$.

Furthermore notice that in contrast to linear elements, quadratic elements have zero phase velocity for $2\,\Delta x$ waves, on a uniform grid. On a non-uniform grid, however, both element types have zero phase velocity for $2\,\Delta x$ waves. In actual simulations (Lynch and Gray, [8]) both element types show good suppression of short wavelength noise. Therefore the Fourier analysis phase portrait does not, in a simple way, give information about ability to suppress short wavelength noise although it does provide information about which wavelengths have accurately modeled phase velocities.

Applications with Even Node Spacing and Variable Bathymetry

In the simulation of field problems one typically encounters situations where the effects of variable bathymetry are important. In this section, bathymetry which varies in an alternating manner from node to node on a regular grid is incorporated into the numerical approximation of Eq. (5). Three different spatial approximations will be examined, LL, LQ, and FD2. The grids for these methods appear in Fig. 4. Note that for the LL and LQ methods, the depths are specified at the nodes whereas for the FD2 approach, the depths are specified midway between the nodes.

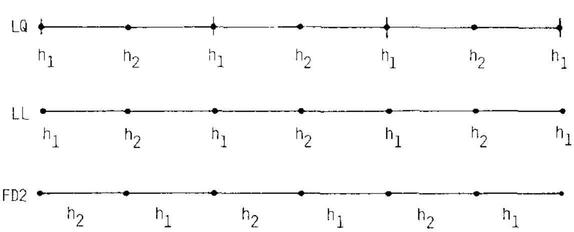

FIG. 4. Grid for numerical schemes analyzed with variable bathymetry.

TABLE VI

Stability Criteria for Explicit Wave Equation Methods on a Uniform One-Dimensional Grid with Variable Bathymetry

Spatial approximation	Second-order finite difference (FD2)	Lumped linear finite element (LL)	Lumped quadratic finite element (LQ)	
Condition	$\bar{H}=\dfrac{h_1+h_2}{2}$	$\bar{H}=\dfrac{h_1+h_2}{2}$	$\dfrac{h_2}{h_1}<\dfrac{5}{2}$	$\dfrac{h_2}{h_1}>\dfrac{5}{2}$
Stability criterion	$1>\dfrac{g\bar{H}\,\Delta t^2}{\Delta x^2}$ and $\tau\,\Delta t>0$	$1>\dfrac{g\bar{H}\,\Delta t^2}{\Delta x^2}$ and $\tau\,\Delta t>0$	$\dfrac{2}{3}>\dfrac{gh_1\,\Delta t^2}{\Delta x^2}$ and $\tau\,\Delta t>0$	$4>\dfrac{g\,\Delta t^2}{\Delta x^2}(h_1+2h_2)$ and $\tau\,\Delta t>0$

Stability criteria for these methods are displayed in Table VI. In general, the stability depends on some weighted combination of the two different bathymetries. The λ-values for the different schemes are displayed in Table VII.

Application to a Rectangular Grid

Although 1-dimensional analyses of stability and accuracy provide insight into the behavior of a numerical scheme, it is also useful to consider some simple 2-

TABLE VII

λ Values for Equi-Spaced Grid Schemes with Variable Bathymetry

Spatial approximation	ω
Second-order finite difference (FD2)	$1\pm\dfrac{\sqrt{(h_1-h_2)^2+4h_1h_2c^2}}{h_1+h_2}$
Lumped linear finite element (LL)	$1\pm\dfrac{\sqrt{(h_1-h_2)^2+4h_1h_2c^2}}{h_1+h_2}$
Lumped quadratic finite element (LQ)	$[3h_1(1+c^2)+2h_2(1-c^2)$ $\pm\sqrt{(h_1[3c^2-1]+2h_2[1-c^2])^2+32h_1^2c^2}]/[2(h_1+h_2)]$

Note. $\lambda=(1-\mathscr{C}_*^2\omega\pm\sqrt{(1-\mathscr{C}_*^2\omega)^2-(1+F)(1-F)})/(1+F)$, $\mathscr{C}_*^2=g(h_1+h_2)\,\Delta t^2/2\,\Delta x^2$, $F=\tau\,\Delta t/2$, $c=\cos(\sigma\,\Delta x)$.

TABLE VIII

Stability Criteria for Explicit Wave Equation Methods Using Different Spatial Discretizations on a Uniform Two-Dimensional Grid with Constant Bathymetry

Spatial approximation	Two-dimensional second order finite difference (FD2)	Two-dimensional lumped linear finite element (LL)	Two-dimensional lumped quadratic finite element (LQ)
Stability criteria	$1 > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$1 > \mathscr{C}^2$ and $\tau\,\Delta t > 0$	$\frac{2}{3} > \mathscr{C}^2$ and $\tau\,\Delta t > 0$

dimensional applications. These derivations are more tedious than the 1-dimensional analyses yet they provide some insight into the more complex dependence of stability on the grid spacings in the x and y directions. Although inclusion of the Coriolis forces in a 2-dimensional analysis is interesting (see, e.g., Kinnmark and Gray [5]), in the interest of simplicity this term will be neglected. Thus Eq. (5) will be analyzed with $f = 0$. A rectangular 2-dimensional grid will be examined in conjunction with lumped linear and quadratic finite elements. Also, results will be presented for the second-order accurate finite difference method.

The analysis of stability proceeds along the same lines as for the 1-dimensional cases. The degree of the resulting stability polynomial is, however, considerably higher. The successful factorization of this into second-degree polynomials relies upon factorization formulas for certain determinants presented in Kinnmark [4]. The possibility of polynomial instability is eliminated if $\tau\,\Delta t > 0$, while exponential

TABLE IX

λ Values for Two-Dimensional Equi-spaced Grid Schemes

Spatial approximation	ω_x, ω_y
Second-order finite difference (FD2)	$\omega_x = 1 - c_x$ $\omega_y = 1 - c_y$
Lumped linear finite element (LL)	$\omega_x = 1 - c_x$ $\omega_y = 1 - c_y$
Lumped quadratic finite element (LQ)	$\omega_x = \frac{1}{4}(5 + c_x^2 \pm \sqrt{(1 + c_x^2)^2 + 32c_x^2})$ $\omega_y = \frac{1}{4}(5 + c_y^2 \pm \sqrt{(1 + c_y^2)^2 + 32c_y^2})$

Note. $\lambda = (1 - \mathscr{C}_x^2\omega_x - \mathscr{C}_y^2\omega_y \pm \sqrt{(1 - \mathscr{C}_x^2\omega_x - \mathscr{C}_y^2\omega_y)^2 - (1 + F)(1 - F)})/(1 + F)$, $\mathscr{C}_x = \sqrt{gh}\,\Delta t/\Delta x$, $\mathscr{C}_y = \sqrt{gh}\,\Delta t/\Delta y$, $F = \tau\,\Delta t/2$, $c_x = \cos(\sigma_x\,\Delta x)$, $c_y = \cos(\sigma_y\,\Delta y)$.

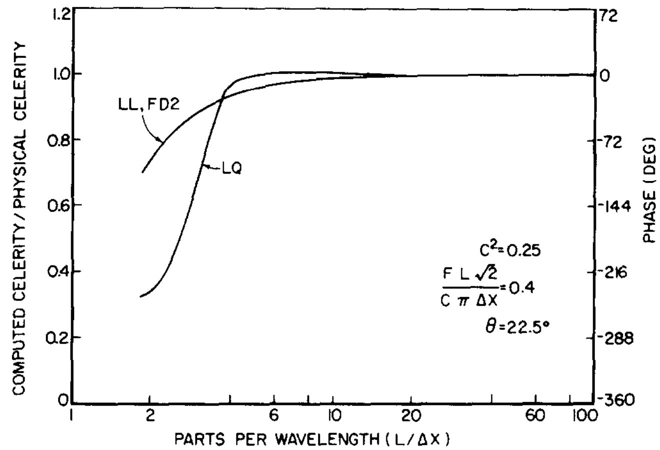

FIG. 5. Phase of the propagation factor for schemes of Table VIII in propagation direction of 22.5° from the x axis.

instability is avoided through satisfaction of the Courant number constraint. The stability constraints obtained appear in Table VIII. Notice that all the stability constraints in this table have the same functional form as their 1-dimensional counterparts with the 1D Courant number squared replaced by the sum of the squares of the Courant numbers in the x and y directions.

For the 2-dimensional cases, the functional form of λ is the same as with the 1D cases. Values of ω are tabulated in Table IX. For simplicity, the accuracy analysis is carried out with $\Delta x = \Delta y$. In the two spatial dimensions, the Fourier analysis can be

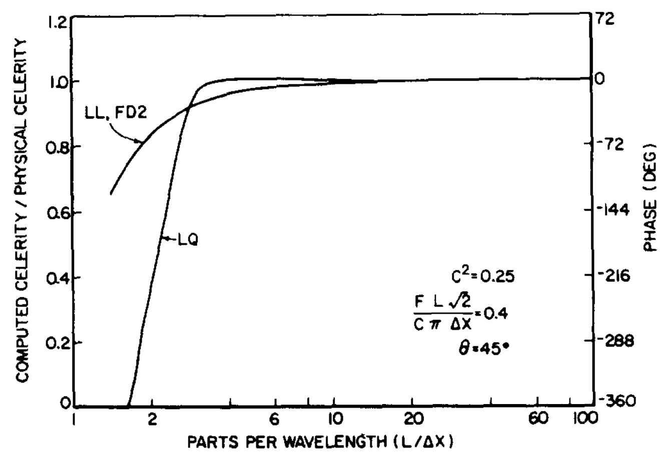

FIG. 6. Phase of the propagation factor for schemes of Table VIII in propagation direction of 45° from the x axis.

carried out for every possible propagation direction for a wave through the mesh. It is sufficient, however, to study the range of directions $0^\circ \leqslant \theta \leqslant 45^\circ$; where θ is the angle between the x axis and the direction of propagation, because every other direction is equivalent to some direction in this range for the square grid. Because $\theta = 0^\circ$ is the same as the 1-dimensional case, the angles $\theta = 22.5^\circ$ and $\theta = 45^\circ$ are selected for study here. To be able to assess the properties of the short wavelength propagation, it is necessary to know the shortest wavelengths which can be represented in any given direction θ. In Kinnmark [4] this is shown to be $2\,\Delta x \cos\theta$ in the $0 \leqslant \theta \leqslant 45^\circ$ range.

The propagation properties of the 2-dimensional schemes are presented in Figs. 5 and 6. It can be seen that the shortest wavelengths for the quadratic elements at $\theta = 0^\circ$ and 45° are not propagated; however, the minimum wavelength for $\theta = 22.5^\circ$ is propagated with this element. The lumped linear element and second-order finite difference scheme, on the other hand, do propagate the minimum wavelength in all three directions. For all cases the amplitude behavior is similar with $|\lambda|$ defined by Eq. (29) for all propagating waves. The magnitudes of λ in Table IX, for the case of negative sign of the quantity under the radical sign, again satisfy (29) independently of Courant number, wavelength, and scheme. Therefore the moduli of the propagation factors essentially contain the same information as in the 1-dimensional case and are therefore not shown.

Conclusions

Six different spatial approximations of the wave equation were compared on a uniform 1-dimensional mesh. All schemes investigated require a positive (non-zero) friction to avoid polynomial instability. Quadratic Lagrangian finite elements, consistent and lumped, were shown to have more severe stability constraints than corresponding linear elements. The fourth-order accuracy five-point finite difference scheme was shown to have a more restrictive stability constraint than the standard three-point, second-order finite difference scheme.

For a simple grid consisting of two alternating mesh widths it was shown that linear elements as well as second- and fourth-order finite differences have a stability constraint which depends on the geometric mean of the two grid sizes. The lumped quadratic element, however, is stable only when the interior node is located in the mid-half of the element.

A uniform grid with two alternating bathymetry values was shown to have a stability constraint governed by the arithmetic mean of the two depths for second-order finite differences and lumped linear finite elements.

The minimum (non-aliased) wavelength which is representable on the 1-dimensional alternating grid is of length equal to the sum of the two alternate lengths. In two spatial dimensions, the minimum (non-aliased) wavelength possible to represent on a quadratic, orthogonal uniform grid was shown to be anisotropic and equal to 2 $\Delta x \cos \theta$, where θ is the angle between the direction of propagation of the wave and the nearest coordinate axis of the grid.

For propagating waves, the amplitudes of the propagation factors were shown to be identical and independent of spatial approximation for moderately sized friction-time-step products and Courant number. This observation is valid for one and two spatial dimensions on a uniform grid as well as on a 1-dimensional grid with alternating mesh widths.

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